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Note on the spanwise quadrature in lifting-wing calculations

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SUMMARY

An improvement concerning the computational economy in lifting-wing calculations is given with applications to Weissinger's lifting-line theory.

1. Introduction

In lifting-line and lifting-surface theory the spanwise integrations are usually carried out with the aid of interpolation functions resulting from the theory of orthogonal polynomials (see e.g. [1] and [2]). The aim of the present note is to state an improvement concerning the computational economy for the method of [2] and to point out alternative methods to [1] and [2] with possible advantages in special cases. Applications to Weissinger's lifting-line theory will be shown.

2. Analysis

Integrals of the type

$$\oint_{-1}^{+1} \frac{\gamma(\eta')}{(\eta - \eta')^2}\, d\eta' \tag{1}$$

and similar ones with $\gamma(\eta)$ representing the spanwise loading are evaluated in [2] by approximating γ over a half span ($0 \leq \eta \leq 1$ or $-1 \leq \eta \leq 0$) in the following manner:

$$\gamma(\vartheta) = \sum_{i=1}^{m+1} \gamma(\vartheta_i) h_i(\vartheta), \tag{2}$$

where $\eta = \cos^2 \vartheta$. From the theory of orthogonal functions optimal stations are derived

$$\vartheta_i = \frac{i\pi}{2m+3}, \qquad i = 1, 2, \ldots, m+1, \tag{3}$$

and also corresponding interpolation functions (so-called station functions)

$$h_i(\vartheta) = \frac{4}{2m+3} \sum_{\mu=0}^{m} \sin(2\mu+1)\vartheta_i \sin(2\mu+1)\vartheta \tag{4}$$

which are equal to one at ϑ_i and vanish at all other ϑ_j. Using this interpolation procedure for γ all integrals in lifting-surface theory like eq. (1) or other type have been evaluated to generate the coefficient matrix for the unknowns $\gamma_i = \gamma(\vartheta_i)$. The disadvantage of this treatment is the fact that each coefficient consists of a sum due to the summation in eq. (4). Therefore the computer time for the generation of the matrix increases proportional to $(m+1)^3$, which implies considerable expense for large values of m in contrast to the method of Multhopp [1].

To avoid this disadvantage it is attempted now to utilize a summation form for the station functions (instead of eq. (4))

$$h_i(\vartheta) = \frac{\sin(2m+1)\vartheta_i}{\cos^2\vartheta - \cos^2\vartheta_i}\,\frac{\sin(2m+3)\vartheta}{2m+3} \tag{5}$$

which is also given in [2] but not used for evaluation of the integrals. When eqs. (2) and (5) are introduced in eq. (1), the resulting integrations can be carried out analytically after decomposing the integrand into partial fractions and applying addition formulas for circular functions. It follows then for a first part of the integral (1) at the stations η_ν ($\eta_\nu > 0$):

$$\int_0^1 \frac{\gamma(\eta')d\eta'}{(\eta_\nu - \eta')^2} = \sum_{i=1}^{m+1} \gamma_i b_{\nu i} \tag{6}$$

with the coefficients

$$b_{\nu i} = \frac{\sin(2m+1)\vartheta_i}{2m+3}\left[\frac{\dfrac{(-1)^m}{\cos^2\vartheta_\nu} + (2m+3)I_{m+1}(\vartheta_\nu)}{\cos^2\vartheta_\nu - \cos^2\vartheta_i} + \right.$$
$$\left. + \frac{1}{2}\,\frac{I_{m+2}(\vartheta_i) - I_m(\vartheta_i)}{(\cos^2\vartheta_i - \cos^2\vartheta_\nu)^2} - \frac{1}{2}\,\frac{I_{m+2}(\vartheta_\nu) - I_m(\vartheta_\nu)}{(\cos^2\vartheta_i - \cos^2\vartheta_\nu)^2}\right], \tag{6a}$$

except for $b_{\nu\nu}$. The integrals $I_k(\vartheta)$ and their recurrence relations are already given in [2]. Treating the special coefficient $b_{\nu\nu}$ in a similar manner, we can express it by introduction of a further integral type also calculable by a recurrence relation. However, the utilization of the summation form (4) in this special case does not imply an essential disadvantage and is preferred in the following. Hence the coefficient $b_{\nu\nu}$ reads (see also [2]):

$$b_{\nu\nu} = -\frac{4}{2m+3}\sum_{\mu=0}^{m} \sin(2\mu+1)\vartheta_\nu\left(\frac{(-1)^\mu}{\cos^2\vartheta_\nu} - (2\mu+1)I_\mu(\vartheta_\nu)\right). \tag{6b}$$

Similarly we get (for $\eta_\nu > 0$ and symmetric loading)

$$\int_{-1}^0 \frac{\gamma(\eta')d\eta'}{(\eta_\nu - \eta')^2} = \int_0^1 \frac{\gamma(\eta')d\eta'}{(\eta_\nu + \eta')^2} = \sum_{i=1}^{m+1} \gamma_i \overline{b_{\nu i}} \tag{7}$$

with

$$\overline{b_{vi}} = \frac{\sin(2m+1)\vartheta_i}{2m+3}\left[\frac{\dfrac{(-1)^m}{\cos^2\vartheta_v} + (2m+3)J_{m+1}(\vartheta_v)}{\cos^2\vartheta_v + \cos^2\vartheta_i} + \right.$$

$$\left. + \frac{1}{2}\frac{I_{m+2}(\vartheta_i) - I_m(\vartheta_i)}{(\cos^2\vartheta_v + \cos^2\vartheta_i)^2} + \frac{1}{2}\frac{J_{m+2}(\vartheta_v) - J_m(\vartheta_v)}{(\cos^2\vartheta_v + \cos^2\vartheta_i)^2}\right]. \tag{7a}$$

The integrals $J_k(\vartheta)$ and their recurrence relations are also given in [2]. Because of a numerical instability in the recurrent calculations of J_k, eq. (7a) has to be replaced in some cases by an expression corresponding to the form below (eq. (8)) as justified in [2].

Further, in Weissinger's lifting-line theory integrals of the form

$$\int_0^1 f(\eta')d\eta' = \sum_{i=1}^{m+1} d_i f(\eta_i); \quad \int_{-1}^0 f(\eta')d\eta' = \sum_{i=1}^{m+1} d_i f(\eta_i) \tag{8}$$

are needed with $f(\eta')$ representing the behavior of $\gamma(\eta')$ within the interval. Employing again the station functions in the form of eq. (5) instead of eq. (4) we obtain the coefficients

$$d_i = \frac{\sin(2m+1)\vartheta_i}{2(2m+3)}(I_{m+2}(\vartheta_i) - I_m(\vartheta_i)). \tag{8a}$$

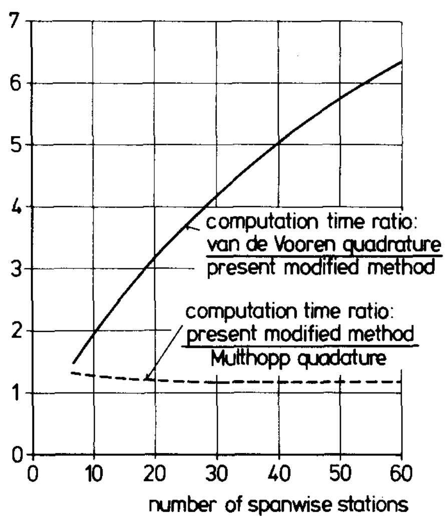

Figure 1. Comparison of computation time for different quadrature methods applied to Weissinger's lifting line theory.

Figure 1 shows a comparison of computer time (CPU-time) for generating and solving the resulting system of linear equations (solution with the aid of a Gaussian elimination process). For higher numbers of spanwise stations the modified procedure presented above saves considerable amounts of computation time in relation to the original method [2] since the time for generating the coefficient matrix increases only proportional to $(m+1)^2$. By introduction of this new procedure no loss of accuracy could be observed. Compared with Multhopp's quadrature [1] method with the same number of stations per half span the calculation time increases only by about 20 percent while all advantages of van de Vooren's method discussed in [2] are maintained.

Moreover, in lifting-surface theory also the integrals

$$\oint_{-1}^{+1} \frac{f(\eta')}{\eta_v - \eta'} d\eta' \quad \text{and} \quad \int_{-1}^{+1} f(\eta') \ln|\eta_v - \eta'| d\eta' \tag{9a, b}$$

occur. The first one can be treated in a similar manner as integral (1). The second one includes a logarithmic singularity within the integrand for $\eta' = \eta_v$ which can be splitted by setting

$$\int_0^1 f(\eta') \ln|\eta_v - \eta'| d\eta' = f(\eta_v) \int_0^1 \ln|\eta_v - \eta'| d\eta' + \\ + \int_0^1 (f(\eta') - f(\eta_v)) \ln|\eta_v - \eta'| d\eta'. \tag{9c}$$

The first integral on the right-hand side of eq. (9c) can be evaluated analytically while in the second term the quadrature can be carried out in accordance to eqs. (8) and (8a) without a noticeable loss of accuracy as shown in [4] for an analogous case. Therefore the method is also applicable in lifting-surface theory.

Furthermore, two methods alternative to Multhopp's and van de Vooren's procedure can be stated which make use of Chebyshev polynomials of the first kind in η instead of the second kind, i.e. cosine series in ϑ instead of sine series. The condition of vanishing γ at the wing tip can be taken into account by splitting in the following manner

$$\gamma(\eta) = f(\eta)\sqrt{1 - \eta^2} \quad \text{or} \quad \gamma(\eta) = f(\eta)\sqrt{1 - \eta}. \tag{10a, b}$$

A procedure analogous to [2] yields for $f(\eta)$ the interpolation functions

$$h_i(\vartheta) = \frac{2(-1)^i \sin \vartheta_i}{(m+1) \cos m\vartheta_i} \left(\frac{1}{2} + \sum_{\mu=1}^{m} \cos \mu\vartheta_i \cos \mu\vartheta \right) \\ = \frac{(-1)^i \sin \vartheta_i \cos (m+1)\vartheta}{(m+1)(\cos \vartheta - \cos \vartheta_i)} \tag{11}$$

with

$$\eta = \cos \vartheta; \qquad \vartheta_i = \frac{\pi(2i+1)}{2(m+1)} \quad \text{for } i = 0, 1, 2, \ldots, m,$$

for a quadrature over the whole span (corresponding to eq. (10a)) and

$$h_i(\vartheta) = \frac{2(-1)^i \sin 2\vartheta_i}{(m+1) \cos 2m\vartheta_i} \left(\frac{1}{2} + \sum_{\mu=1}^{m} \cos 2\mu\vartheta_i \cos 2\mu\vartheta \right) \\ = \frac{(-1)^i \sin 2\vartheta_i \cos 2(m+1)\vartheta}{2(m+1)(\cos^2 \vartheta - \cos^2 \vartheta_i)} \tag{12}$$

with

$$\eta = \cos^2 \vartheta; \qquad \vartheta_i = \frac{\pi(2i+1)}{4(m+1)} \quad \text{for } i = 0, 1, 2, \ldots, m,$$

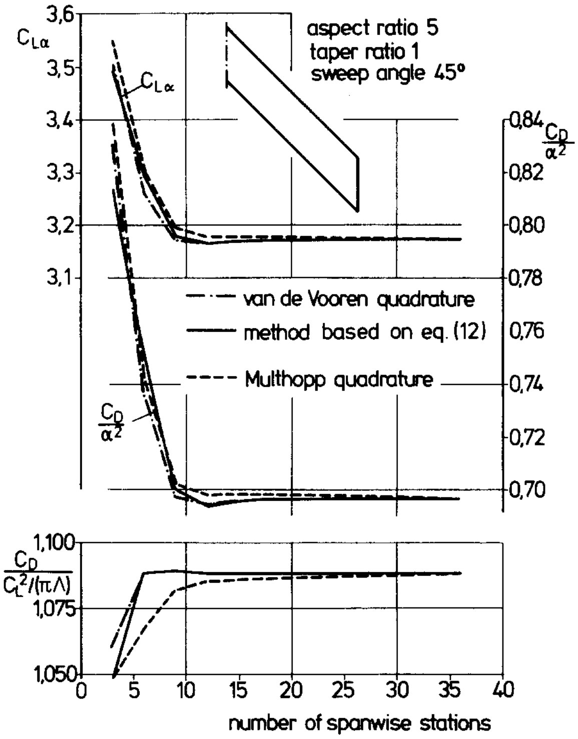

Figure 2. Comparison of convergence for different quadrature methods applied to Weissinger's lifting line theory (Λ = aspect ratio).

for a quadrature over the half span (corresponding to eq. (10b)). The derivation of eq. (12) is given in detail in [5]. The application of this interpolation methods to lifting-line theory leads to only slightly more complicated formulae compared with those given above. The accuracy is as good as in the corresponding cases [1, 2] and the computation time equals almost exactly that for the "advanced van de Vooren method" discussed above.

Figure 2 shows for an arrow wing the comparison of the convergence behavior with increasing number of stations for three quadrature methods, namely the modified van de Vooren method (see eqs. (3) to (8a); convergence identical with van de Vooren's original procedure), the method connected with eq. (12), and Multhopp's method. Only small differences can be observed between the results of the different procedures for the lift gradient $C_{L\alpha}$ and for the induced drag C_D/α^2 but a more pronounced variation exists for the drag parameter $C_{Di}/(C_A^2/(\pi\Lambda))$. Multhopp's method is inferior to both other methods since the behavior of the spanwise loading near $\eta = 0$ for arrow wings cannot be described as well as by both other methods (see also [2]). But both other methods are equivalent with respect to economy and accuracy and it depends on other criteria which of them should be chosen for a special case.

However, advantages of the methods connected with eqs. (11) and (12) are the following facts:

1. The utmost spanwise station is situated closer to the wing tip since its distance in ϑ from the tip is only a half interval.

2. The accuracy in the vicinity of the tips is increased. This follows from the approximation property of orthogonal polynomials φ_n of degree n ([6]):

$$\int_a^b g(x)\,(f(x) - \varphi_n(x))^2\,dx = \text{minimum} \tag{13}$$

with $f(x)$ denoting the function to be approximated within the interval $a \leq x \leq b$ and $g(x)$ being a weighting function which results from the orthogonality relation. E.g. the approximation functions (12) correspond to

$$g(\eta) = \frac{1}{\sqrt{\eta(1 - \eta)}} \quad \text{for } 0 \leq \eta \leq 1$$

instead of

$$g(\eta) = \sqrt{\frac{1 - \eta}{\eta}}$$

for van de Vooren's method and therefore the accuracy is increased at the tip ($\eta \to 1$) with regard to eq. (13).

3. These methods are capable to fulfil another boundary condition at the wing tip, namely nonvanishing $\gamma(\eta = 1)$ which e.g. is valid at endplates.

Hence for special purposes these methods may be preferred to the classical ones [1, 2].

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